DERIVING THE DEMAND CURVE ASSUMING THAT THE MARGINAL UTILITY FUNCTIONS ARE LINEAR

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Abstract: This paperwork derives the function of demand depending on price assuming that the marginal utility functions are linear. The main idea is that we can derive the demand function starting from the equimarginal principle. Starting from the equimarginal principle and from the budget equation and assuming that the marginal utility functions are linear we can derive the demand function.

INTRODUCTION

The economic literature explains that between demand and price exists an inverse connection. More than that, this connection is demonstrated qualitatively starting from the equimarginal principle. This paperwork tries to derive a quantitative connection between demand and price starting form the equimarginal principle, assuming that the marginal utilities are linear functions.

MATERIALS AND METHODS

As it is known in economic theory, the consumer maximizes total utility when the following conditions are verified simultaneously:

\[
\frac{u_a(Q_a)}{P_a} = \frac{u_b(Q_b)}{P_b} = \ldots = \frac{u_n(Q_n)}{P_n}
\]

\[P_a \cdot Q_a + P_b \cdot Q_b + \ldots + P_n \cdot Q_n = B
\]

Where \(Q_a, Q_b, \ldots, Q_n\) are non-negative and are \(P_a, P_b, \ldots, P_n\) no negative.

As one can observe, the equimarginal principle leads us to the following system of equations:

\[
\begin{align*}
\frac{u_a(Q_a)}{P_a} &= \frac{u_b(Q_b)}{P_b} \\
\hdots \\
\frac{u_n(Q_n)}{P_n} &= \frac{u_n(Q_n)}{P_n}
\end{align*}
\]

\[P_a \cdot Q_a + P_b \cdot Q_b + \ldots + P_n \cdot Q_n = B
\]
The system above has \( n \) equations and \( n \) unknowns. From the \( n \) equations, \( n-1 \) equations results from the row of equimarginal equations, and the \( n \)-th equation is the budget equation.

When marginal utility of each good is a linear function, the system above becomes a linear system of \( n \) equations with \( n \) unknowns.

If the marginal utility functions are linear, than they may be expressed in the following manner:

\[
\begin{align*}
u_a(Q_a) &= s_a + t_a \cdot Q_a & \text{Where } s_a > 0 \text{ and } t_a < 0 \\
u_b(Q_b) &= s_b + t_b \cdot Q_b & \text{Where } s_b > 0 \text{ and } t_b < 0 \\
& \vdots & \\
u_n(Q_n) &= s_n + t_n \cdot Q_n & \text{Where } s_n > 0 \text{ and } t_n < 0
\end{align*}
\]

Under these conditions the system of \( n \) linear equations with \( n \) unknowns above can be written under the following form:

\[
\begin{align*}
\frac{s_a + t_a \cdot Q_a}{P_a} &= \frac{s_b + t_b \cdot Q_b}{P_b} \\
& \vdots \\
\frac{s_a + t_a \cdot Q_a}{P_a} &= \frac{s_b + t_b \cdot Q_b}{P_b} \\
P_a \cdot Q_a + P_b \cdot Q_b + \ldots + P_n \cdot Q_n &= B
\end{align*}
\]

The solution of this system of equation is easier than the general case of a system of \( n \) equations with \( n \) unknowns. This solution can be done in the following manner:

One express \( Q_a \) function of \( Q_a \) from the first equation, \( Q_c \) function of \( Q_a \) from the second equation, and so on until the last equation where we express \( Q_n \) function of \( Q_a \).

One replace \( Q_a, Q_c, \ldots, Q_n \) in the budget equation, resulting a linear equation with one unknown - \( Q_a \). Solving this equation we obtain \( Q_a \).

One replace \( Q_a \) in the first \( n-1 \) equations and this manner one obtain the values of \( Q_b, Q_c, \ldots, Q_n \).

One special case which may appear is that in which one or more of the quantities \( Q_b, Q_c, \ldots, Q_n \) are negative. The initial condition is that all quantities are no negative.

Let us suppose that one of the quantities which verify the system of equations is negative. Let us suppose that this is \( Q_n \). In this case we will assign to \( Q_n \) value zero and we will solve the system of \( n-1 \) equations with \( n-1 \) unknowns which follows:

\[
\begin{align*}
\frac{s_a + t_a \cdot Q_a}{P_a} &= \frac{s_b + t_b \cdot Q_b}{P_b} \\
& \vdots \\
\frac{s_a + t_a \cdot Q_a}{P_a} &= \frac{s_m + t_m \cdot Q_m}{P_m} \\
P_a \cdot Q_a + P_b \cdot Q_b + \ldots + P_n \cdot Q_n &= B
\end{align*}
\]
The solution of this system will be done analogue to that with \( n \) equations with \( n \) unknowns and will lead us to the solutions \( Q_a, Q_b, \ldots, Q_m \).

RESULTS AND DISCUSSIONS

Now we will derive the demand curve for good “a” in two concrete cases in which we take into consideration only two goods. From the shape point of view, the graphic of demand for good “a” is the same indifferent how many goods we take into consideration. The mathematical calculus is analogue but becomes more laborious.

**Example 1**

\[
\begin{align*}
  u_a(Q_a) &= 100 - 2 \cdot Q_a \\
  u_b(Q_b) &= 120 - 3 \cdot Q_b \\
  P_a &= \text{a parameter which may take positive values} \\
  P_b &= 9; \ B = 210
\end{align*}
\]

The system of equations which result from the equi marginal principle is:

\[
\begin{cases}
  100 - 2 \cdot Q_a = 120 - 3 \cdot Q_b \\
  P_a \cdot Q_a + 9 \cdot Q_b = 210
\end{cases}
\]

With unknowns \( Q_a \) and \( Q_b \)

The solution of the equations system leads to the following solutions:

\[
\begin{align*}
  Q_a &= \frac{2700 - 150 \cdot P_a}{P_a^2 + 54} \\
  Q_b &= \frac{40 \cdot P_a^2 - 300 \cdot P_a + 1260}{P_a^2 + 54}
\end{align*}
\]

\[
\begin{align*}
  Q_a &= 150 \cdot \frac{18 - P_a}{P_a^2 + 54} \\
  Q_b &= 20 \cdot \frac{2 \cdot P_a^2 - 15 \cdot P_a + 63}{P_a^2 + 54}
\end{align*}
\]

The condition of no negativity for the quantity \( Q_a \) leads us to:

\[
150 \cdot \frac{18 - P_a}{P_a^2 + 54} \geq 0 \quad \Leftrightarrow \quad P_a \leq 18
\]

The condition of no negativity for quantity \( Q_b \) leads us to:

\[
20 \cdot \frac{2 \cdot P_a^2 - 15 \cdot P_a + 63}{P_a^2 + 54} \geq 0 \quad \text{Which is equivalent to:}
\]

\[
2 \cdot P_a^2 - 15 \cdot P_a + 63 \geq 0 \quad \text{Which is verified for each } P_a \text{ because:}
\]

\[
\Delta = 15^2 - 4 \cdot 2 \cdot 63 = -279 < 0
\]

The graphic of demand for good “a” depending on price is the following:
Example 2

\[ u_a(Q_a) = 100 - 2 \cdot Q_a \]
\[ u_b(Q_b) = 120 - 3 \cdot Q_b \]

\( P_a \) is a parameter which may take positive values

\( P_b = 12 ; \quad B = 120 \)

The system of equations which results from the equimarginal principle is:

\[
\begin{align*}
100 - 2 \cdot Q_a &= 120 - 3 \cdot Q_b \\
\frac{P_a}{12} &= Q_a \\
P_a \cdot Q_a + 12 \cdot Q_b &= 120
\end{align*}
\]

With the unknowns \( Q_a \) and \( Q_b \)

Solving the equation system we obtain:

\[
\begin{align*}
Q_a &= \frac{4800 - 360 \cdot P_a}{P_a^2 + 96} \\
Q_b &= \frac{120 \cdot P_a^2 - 1200 \cdot P_a + 2880}{3 \cdot (P_a^2 + 96)} \quad \Leftrightarrow \quad Q_a = 120 \cdot \frac{40 - 3 \cdot P_a}{P_a^2 + 96} \\
Q_b &= 40 \cdot \frac{P_a^2 - 10 \cdot P_a + 24}{P_a^2 + 96}
\end{align*}
\]

The condition of no negativity for \( Q_a \) leads us to:

\[ Q_a = 120 \cdot \frac{40 - 3 \cdot P_a}{P_a^2 + 96} \geq 0 \quad \Leftrightarrow \quad P_a \leq 13, (3) \]
The condition of no negativity for \( Q_a \) leads us to:

\[
40 \cdot \frac{P_a^2 - 10 \cdot P_a + 24}{P_a^2 + 96} \geq 0
\]

Which is equivalent to

\[
P_a^2 - 10 \cdot P_a + 24 \geq 0
\]

Which is verified for \( P_a \in \left[0;4\right] \cup \left[6;13\right] \).

For \( P_a \in \left[4;6\right] \) only one good will be purchased, good “a”.

The demand function for good “a” will be:

\[
Q_a = \frac{120}{P_a}
\]

The graphic of demand for good “a” function of price will depend on the interval which contains \( P_a \). It will be the following one:

\[
\text{CONCLUSIONS}
\]

OBS. In order to obtain graphics of this type one has to accomplish the following condition:

\[
\frac{s_b}{-t_b} \cdot P_b + \frac{s_c}{-t_c} \cdot P_c + \ldots + \frac{s_n}{-t_n} \cdot P_n > B
\]

This condition says that if we allocate the budget for all the other goods except good “a” we will not reach to the total satisfaction of all other needs, needs that are satisfied with goods “b”, “c”, “n”,

As we can see in both situations took as example, the function price - demand is decreasing. In the second case the individual demand function is not derivable on the entire domain of definition.

\[
\text{REFERENCES}
\]