CONSIDERATIONS ON LINEAR CONFORMAL TRANSFORMATION

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Abstract. *The paper refers to the transformation of coordinates from a rectangular coordinate system into another system. Depending on the accuracy required and distance between points, topographical methods of transformation are presented. It also shows, the calculation of linear transformation parameters: translation in the direction of axes, rotation of the two systems and module scale. The paper concludes with an application for conversion of coordinates from a local system in the national Stereo 1970.*

Keywords: coordinate transformations, coordinate conversions, coordinate trans-calculation

INTRODUCTION

In areas of interest where there is not a proper density of points in stereographical 1970 projection, but there are points in a local projection or other projection systems, it is necessary to transform these systems coordinates in the national Stereo 1970.

By coordinate transformation means the mathematical operation by which coordinates, known in a particular system are calculated in another system. If coordinate transformations are made between two systems that have the same datum, transformation is named conversion. If the two systems have different datums coordinate transformation is named trans-calculation (Moldoveanu 2002).

Depending on the required accuracy and distance between points, to solve the problem, it can be chosen a topographical or a geodetic solution. Topographic solution is chosen for small distances, where the curvature effect and the variance of linear deformations are neglected.

Geodesic method is chosen for large distances between the points of II, III and IV order of geodetic networks, in which case it will take into account the effect of curvature and the variation of linear deformations.

MATERIAL AND METHOD

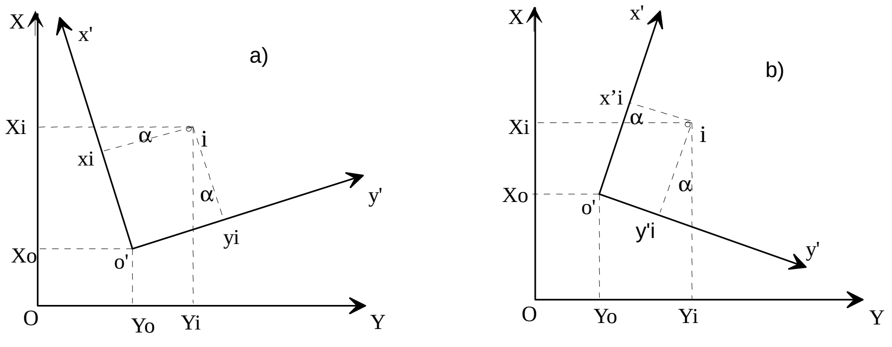

Fig. 1. Coordinates transformation
a) counterclockwise rotation; b) clockwise rotation

General formulas of transformation are similar to those of analytic geometry and can easily be written, following figure 1.

Taking the coordinates of point i (Xi, Yi) in the xOy national coordinates axes system and respectively i(xi, yi) in local coordinate system $x'o'y'$, for a flat rectangular transformation of the local system (system I) in the national system (system II) relations apply:

$$\begin{aligned} X_i &= X_{o'} + x_i^{'} \cos\alpha \pm y_i^{'} \sin\alpha \\ Y_i &= Y_{o'} \mp x_i^{'} \sin\alpha + y_i^{'} \cos\alpha \end{aligned} \qquad (1)$$

where:

Xo 'Yo' - plane coordinates of the local system origin in the national system. This coordinates, in general unknown, represent the translation of the local system compared to the national system.

α - the rotation angle formed by the axes of the two coordinate systems.

Equation (1) is rotation to the left of the local system (Fig. 1. a), without taking into account the sign of rotation angle.

Assuming that the angle of rotation between the two systems could be calculated with:

$$\alpha = \theta - \theta' \qquad (2)$$

where: θ - orientation of OX-axis of the national system;

θ' - orientation of $o'x'$-axis of the local system.

Then, for the left rotation of the local system, the angle of rotation is negative. Considering the angle $(-\alpha)$ in quadrant IV and the angle α in quadrant I and taking into account the transformation of trigonometric functions between the two quadrants, the rotation matrix takes the form:

$$R = \begin{pmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad (3)$$

Relation (1), written on matrix form:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} X_{o'} \\ Y_{o'} \end{pmatrix} + \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_i^{'} \\ y_i^{'} \end{pmatrix} \qquad (4)$$

Performing calculations we get:

$$\begin{aligned} X_i &= X_{o'} + x_i^{'} \cos\alpha - y_i^{'} \sin\alpha \\ Y_1 &= Y_{o'} + x_i^{'} \sin\alpha + y_i^{'} \cos\alpha \end{aligned} \qquad (5)$$

Note: In relation (5) takes into account that the angle is negative, so $\cos(-\alpha) = \cos\alpha, \sin(-\alpha) = -\sin\alpha$ and thus that equation (5) becomes identical with (1) fixed geometrically.

When transforming coordinates from a local system in the national system it should be taken into account, in addition to the angle of rotation of the two systems, the linear deformation module "k".

$$k = \frac{[D_{ij}]}{[d_{ij}]} \qquad (6)$$

where

[D_{ij}] - sum of distances in the national system (the coordinates turns in)

[d_{ij}] - sum of distances in the local system (from which transforms the coordinates).

Taking into account the magnitude of deformation equation (8) becomes:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} X_{o'} \\ Y_{o'} \end{pmatrix} + k \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_i^{'} \\ y_i^{'} \end{pmatrix} \qquad (7)$$

Performing calculations we get:

$$\begin{aligned} X_i &= X_{o'} + x_i^{'} k\cos\alpha - y_i^{'} k\sin\alpha \\ Y_i &= Y_{o'} + x_i^{'} k\sin\alpha + y_i^{'} k\cos\alpha \end{aligned} \qquad (8)$$

To convert rectangular coordinates from a local system in a national system should be minimum two points with known coordinates in both systems. It is noted that in (8) are four unknowns: the geodetic coordinates of particular system origin $X_{o'}\, Y_{o'}$, the angle of rotation α and scale module "k" that appear as nonlinear functions $k\cos\alpha, k\sin\alpha$. Calculation of unknowns can be achieved by topographic methods or by the least squares method. It is presented the calculus of transformation parameters by the least squares method. Method applies when more points with coordinates in both systems are known. In equation (8), is considered that the most likely values of the coordinates of the points in the national system are equal with provisional values (Xi, Yi) plus their corrections (V_{Xi}, V_{Yi}), so for "n" points can be written the equation system:

$$\begin{aligned} X_i + V_{Xi} &= X_{o'} + x_i^{'} k\cos\alpha - y_i^{'} k\sin\alpha \\ Y_i + V_{Yi} &= Y_{o'} + y_i^{'} k\cos\alpha + x_i^{'} k\sin\alpha \qquad i = 1, 2 \ldots n \end{aligned} \qquad (9)$$

From where results:

$$\begin{aligned} V_{Xi} &= X_{o'} + x_i^{'} k\cos\alpha - y_i^{'} k\sin\alpha - X_i \\ V_{Yi} &= Y_{o'} + y_i^{'} k\cos\alpha + x_i^{'} k\sin\alpha - Y_i \qquad i = 1, 2 \ldots n \end{aligned} \qquad (10)$$

For linearization of equations (10) the following changes of variables are done: $k\cos\alpha = a,\ \ k\sin\alpha = b,\ \ Xo' = a_0,\ \ Yo' = b_0$ and then the system of "2n" equations with four unknowns can be written as (Danciu 2002):

$$\begin{aligned} V_{X1} &= a_0 + x_1^{'} a - y_1 b - X_1 \\ V_{Y1} &= b_0 + y_1^{'} a + x_1^{'} b - Y_1 \\ V_{X2} &= a_0 + x_2^{'} a - y_2^{'} b - X_2 \\ V_{Y2} &= b_0 + y_2^{'} a + x_2^{'} b - Y_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V_{Xn} &= a_0 + x_n^{'} a - y_n^{'} b - X_n \\ V_{Yn} &= b_0 + y_n^{'} a_1 + x_n^{'} b_1 - Y_n \end{aligned} \qquad (11)$$

On matrix form the system (11) can be written (Ortelecan 2006):

$$AX - l = V \qquad (12)$$

where:

$$A_{(2n,4)} = \begin{pmatrix} 1 & 0 & x_1' & -y_1' \\ 0 & 1 & y_1' & x_1' \\ 1 & 0 & x_2' & -y_2' \\ 0 & 1 & y_2' & x_2' \\ \dots & \dots & \dots & \dots \\ 0 & 1 & y_n' & x_n' \end{pmatrix}, \quad X_{(4,1)} = \begin{pmatrix} a_0 \\ b_0 \\ a \\ b \end{pmatrix}, \quad l_{(2n,1)} = -\begin{pmatrix} -X1 \\ -Y1 \\ -X2 \\ -Y2 \\ \dots \\ -Yn \end{pmatrix}, \quad V_{(2n,1)} = \begin{pmatrix} V_{X1} \\ V_{Y1} \\ V_{X2} \\ V_{Y2} \\ \dots \\ V_{Yn} \end{pmatrix} \qquad (13)$$

We put the minimum condition:

$$[VV]_X + [VV]_Y \rightarrow \text{minim}$$

So it reaches the normal system of equations that takes the form:

$$(A^T A)X - A^T l = 0 \qquad (14)$$

where:

A^T transposed matrix of coefficients

From equation (14) results the matrix of unknowns:

$$X_{(4,1)} = (A^T_{(4,2n)} A_{(2n,4)})^{-1} A^T_{(4,2n)} l_{(2n,1)} \qquad (15)$$

Mean square error of one observation (standard deviation) is calculated with the formula:

$$m_0 = \pm\sqrt{\frac{V^T V}{n-k}} = \pm\sqrt{\frac{V^T V}{2n-4}} \qquad (16)$$

$$V^T V = l^T (E - A(A^T A)^{-1} A^*) l \qquad (17)$$

where:

E - unit matrix.

"V" correction and then $V^T V$ can also be calculated by introducing unknowns calculated by equation (15) in equation (11).

Mean square error of the unknowns is calculated with the relations:

$$\begin{aligned} m_{a_0} &= \pm m_0 \sqrt{Q_{11}} \\ m_{b_0} &= \pm m_0 \sqrt{Q_{22}} \\ m_a &= \pm m_0 \sqrt{Q_{33}} \\ m_b &= \pm m_0 \sqrt{Q_{44}} \end{aligned} \qquad (18)$$

where:

Qii - weighting coefficients, which are found on the main diagonal of the inverse matrix $Q_{xx} = (A^T A)^{-1}$

Using the determined transformation parameters it will calculate the coordinates of other points in the national system known only locally. It will use the relation (19):

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad (19)$$

CASE STUDY

In order to realize the spacial base of a building from Grigorescu district, Cluj-Napoca city, surveying was performed in a local system (Tabel 1) using a Leica TCR 805 total station. Local coordinate system was used due to lack of points from the national system and visibility to these points. To realize the property digital plan in the national coordinate system, the coordinates in the national system Stereo1970 for three common points were determined using the RTK global positioning receiver (Table1).

Table 1

Point	Stereo 1970		Local system	
	X	Y	x	y
101	586271,272	389118,402	580000,000	385000,000
102	586312,844	389079,549	580056,915	385000,000
103	586445,942	389290,121	580010,361	385244,724
44			580058,0924	385001,89
35			579995,1054	385022,055
36			579988,1656	384995,577
37			579994,1699	384990,203
38			580000,1324	385012,839

Doing the calculation from equation (15) we obtain:

$$A = \begin{pmatrix} 1 & 0 & 580000{,}000 & -385000{,}000 \\ 0 & 1 & 385000{,}000 & 580000{,}000 \\ 1 & 0 & 580056{,}915 & -385000{,}000 \\ 0 & 1 & 385000{,}000 & 580056{,}915 \\ 1 & 0 & 580010{,}361 & -385244{,}724 \\ 0 & 1 & 385244{,}724 & 580010{,}361 \end{pmatrix}; \quad l = \begin{pmatrix} 586271{,}272 \\ 389118{,}402 \\ 586312{,}844 \\ 389079{,}549 \\ 586445{,}942 \\ 389290{,}121 \end{pmatrix}$$

$$(A^T A)^{-1} = \begin{pmatrix} 11605893{,}92 & -0{,}008292301 & -13{,}88794395 & 9{,}22031821 \\ 0{,}04566512 & 11605893{,}86 & -9{,}22031821 & -13{,}88794385 \\ -13{,}88794399 & -9{,}220318146 & 2{,}39438E\text{-}05 & -9{,}42105E\text{-}14 \\ 9{,}220318146 & -13{,}88794389 & 1{,}71076E\text{-}14 & 2{,}39438E\text{-}05 \end{pmatrix}$$

$$A^T l = \begin{pmatrix} 1759030{,}058 \\ 1167488{,}072 \\ 1{,}46986E+12 \\ -200829618{,}5 \end{pmatrix}; \quad X = \begin{pmatrix} ao \\ bo \\ a1 \\ b1 \end{pmatrix} = (A^T A)^{-1} A^T l = \begin{pmatrix} -100345{,}4965 \\ 503886{,}3339 \\ 0{,}730569135 \\ -0{,}682822485 \end{pmatrix}$$

From parameters a_1 and b_1 are calculated angle of rotation between the two systems and scale module:

$$\alpha = arc\tan\left(\frac{b}{a}\right) = -47.8502$$

$$m = \frac{a}{\cos\alpha} = \frac{b}{\sin\alpha} = 0.9999889$$

To check the accuracy of determining the transformation parameters were calculated coordinates of common points in the national system with parameters obtained from the calculation and the resulting values are presented in Table 2.

Table 2

Point	Known coordinates		Unknown coordinates		Differences	
	X	Y	X	Y	dx	dy
101	586271,272	389118,402	586271,259	389118,409	0,013	-0,007
102	586312,844	389079,549	586312,839	389079,547	0,005	0,002
103	586445,942	389290,121	586445,931	389290,122	0,011	-0,001

Standard deviation for calculation of transformation parameters is calculated with relation (19) and the values obtained are: m_0 = 21.8 mm, m_a = 0.1 mm, m_b = 0.1 mm.

With determined transformation parameters values are obtained transformed values of local coordinates of the points in the national system by using relation (19), values presented in Table 3.

Table 3

Point	X	Y
44	586314,9899	389080,1237
35	586282,7423	389137,8642
36	586259,5928	389123,2592
37	586260,3095	389115,2328
38	586280,1217	389127,6985

CONCLUSIONS

Plane orthogonal transformation is applied when the transformed points are integrated within the topographical area. Disposition of coordinates axes, in both systems, must keep the same direction. Rotation with more than 90^0 degrees of rotation angle between the two systems leads to a change of the signs in the calculation relations. Estimation of transformation accuracy is achieved with standard deviation, calculated from the difference between initial and transformed coordinates of common points. The more common points the conversion accuracy is better and in this case multiple plane orthogonal transformation is used. The accuracy of transformation parameters calculation of different with small values, depending on the layout in equations of the parameters calculated.

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