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DIFFERENTIAL EQUATIONS AND THEIRS APPLICATION TO THE SOIL MOISTURE STUDY

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Abstract: In this note we make a connection between diffusion equation and equations of Darcy and Richard used in study the soil hydrodynamics.

INTRODUCTION

Most physical phenomena (e.g. fluid dynamics) can be modeled using partial differential equation. A partial differential equation is an equation that contains partial derivatives of unknown function. Two of the most common examples are speeded of milk in a cup of coffee or warming of a rod when at the end of it there is one or two source of heat. The first one study the diffusion of milk in a cup of coffee; the milk concentration depends upon time elapsed by the time when someone put the milk in a coffee and by the place where he pour the milk. The second one study the dissipation of the heat into the rod that depends by the time and by the position of considered point into the rod. Also we can imagine how the water distributes into the soil layers, this phenomenon depend also by the time and space and at the end of this paper we show that it governed by the diffusion equation.

HYDRODYNAMICS OF SOIL WATER AND DIFFUSION EQUATION

The hydrodynamics of the water into soil is a very complex phenomenon. In order to understand this phenomenon soil scientists have made some models for the flow of water into soil. Depending by the parameters that will be considered and the way that we modeling the soil water flow into the soil we can obtain different type of equations. It is important to have equilibrium between the assumptions and the parameters involved in a model. Taking into consideration previous affirmation, we have different type of equations that models the water flow into soil.

1. Darcy's equation The assumptions in this case are:

- soil is saturated with water;
- water is flowing in all pores under a positive pressure head h.

Usually the soil is quasi-saturated with water, but for this case of saturated flow the impact of pores filled with air is not considered.

The soil is paced in a horizontal cylinder connected in both sides with vessel filled with water, maintaining in both side a constant level of water. If the water level in the left side is higher that on the right side the water will flow from left to right. The flux density $q = \frac{V}{At}$ where V is the volumetric overflow, t is time and A is the area of a cross section of the cylinder perpendicular to the direction of flow. Starting from this relation the Darcy have obtained the following experimental law:

$$q = -K_s \nabla H \tag{1}$$

where:

- $H=h+z$ represents the total potential head and z represents the term due to elevation in case when the soil is put in a vertical cylinder;
- K_s represents saturated hydraulic conductivity of the soil.

Other limitation of Darcy's equation is given by the low gradient.

2. Unsaturated flow in rigid soil A rigid soil is that do not change their bulk volume with change of water content [3]. The assumption in this case is that

- pores that are filled with air could resaturate or drain;
- the capillarity effect is not take into consideration.

The experiment in this case considers an unsaturated soil column and we can make an analogy with a flow in a syphon with an installed resistance. The flux density depends upon the hydraulic gradient and is governed by a equation similar to (1):

$$q = K \frac{\Delta H}{L} \tag{2}$$

where L is the soil length and K is unsaturated soil conductivity [LT^{-1}]). Because the soil is not saturated and flow occurs primarily in pores filled with water K is smaller than K_s for the same but saturated soil. From this reason K will be a function of the soil water potential head and equation (2) became

$$q = K(h) \frac{dH}{dz} \tag{3}$$

If we work in 2 or 3 dimensions we have:

$$q = K(h) \nabla H \tag{4}$$

Buckingham was the first that had described the dependence of unsaturated flow upon the potential gradient so equation (3) and (4) are named **Darcy-Buckingham equations**. These type of equations are adequate for describing unsaturated flow only if the soil water content is not changing in time, but this case is very seldom one.

When θ and q are changing in time equation (3) or (4) must be combined with the continuity equation. Continuity equation relates the rate of change in time for θ with the spatial rate of change for q in a smaller elementary soil volume. The result is a non linear equation and even for simply conditions the solution is difficult to find.

The flux density is described by Darcy-Buckingham equation and the rate of filling or emptying of the pores of the soil is described by the equation of continuity. Consider the volume element having the edges of length $\Delta x, \Delta y$ and Δz. The difference between the volume of water flowing into the volume element and volume of water that flowing out is equal to the difference of water content in the element of time Δt. The rate of inflow in x direction is q_x. We assume that the rate of change for q_x is continuous so the rate of outflow is $q_x + \frac{\partial q_x}{\partial x} \Delta x$. The inflow volume is $q_x \Delta y \Delta z \Delta t$ and the outflow volume is $\left(q_x + \frac{\partial q_x}{\partial x} \Delta x \right) \Delta y \Delta z \Delta t$. The difference between inflow and outflow in x direction is:

$$-\frac{\partial q_x}{\partial x} \Delta x \Delta y \Delta z \Delta t \tag{5}$$

The differences between inflow and outflow in y and z direction are:

$$-\frac{\partial q_y}{\partial x}\Delta y\Delta x\Delta z\Delta t \tag{6}$$

$$-\frac{\partial q_z}{\partial x}\Delta z\Delta x\Delta y\Delta t \tag{7}$$

The change in water content for the entire representative volume is the sum of (5), (6) and (7) that means:

$$\frac{\Delta\theta}{\Delta t}\Delta x\Delta y\Delta z\Delta t = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)\Delta x\Delta y\Delta z\Delta t \tag{8}$$

when $t \to 0$ equation (8) became

$$\frac{\partial\theta}{\partial t} = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right) \tag{9}$$

Combining (9) with (3) we have:

$$\frac{\partial\theta}{\partial t} = -\left(\frac{\partial}{\partial x}\left(K(h)\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(K(h)\frac{\partial H}{\partial y}\right)+\frac{\partial}{\partial z}\left(K(h)\frac{\partial H}{\partial z}\right)\right) \tag{10}$$

If we work in one dimension and we assume that the soil is isotropic then

$$\frac{\partial\theta}{\partial t} = -\frac{\partial}{\partial z}\left(K(h)\frac{\partial(h+z)}{\partial z}\right) \Leftrightarrow \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(K(h)\frac{\partial h}{\partial z}\right)+\frac{\partial K}{\partial z} \tag{11}$$

Equations (10) and (11) are called **Richard's equations**.

3. **Water flow in non rigid soils.** When a soil swells or shrinks due to the water content the equations mention above do not work. A theory for the three dimensional case is an open problem. There are some results for one dimensional case [4, 5] and describe wetting of artificially repacked soil column in a laboratory.

4. **Connection to the diffusion equation.** The diffusion equation is a parabolic partial differential equation. The general forms of this equation are:

$$\frac{\partial u}{\partial t} = D(t,x)\frac{\partial^2 u}{\partial x^2} \text{ (one dimensional diffusion equation)} \tag{12}$$

$$\frac{\partial u}{\partial t} = D(t,x)\Delta u \text{ (n dimension diffusion equation)} \tag{13}$$

$$\frac{\partial u}{\partial t} = D(t,x)\Delta u - h\nabla u \text{ (n dimension diffusion – convection equation)} \tag{14}$$

$$\frac{\partial u}{\partial t} = D(t,x)\Delta u + f(t,x) \text{ (15) (diffusion equation with heat source)} \tag{15}$$

In what it follows we emphasize the connection between diffusion equation in forms mentioned above and Darcy's and Richard's equations. The general case is the diffusion equations; in order to show the "equivalence" we must to connect diffusion coefficient, *D*, with hydraulic conductivity *K* [6]. If we use $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial\theta}\frac{\partial\theta}{\partial z}$, the Richard's equation became:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(K(h)\frac{\partial h}{\partial\theta}\frac{\partial\theta}{\partial z}\right)+\frac{\partial K}{\partial\theta}\frac{\partial\theta}{\partial z} \tag{16}$$

The connection between soil water diffusivity *D*, θ and *K* is

$$K(\theta) = D(\theta)\frac{\partial\theta}{\partial h} \tag{17}$$

then equation (16) became:

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(D(\theta)\frac{\partial\theta}{\partial z}\right)+\frac{\partial K}{\partial\theta}\frac{\partial\theta}{\partial z} \tag{18}$$

and represents the Richard's equation in diffusivity form and is in the form of classical mathematical equation (14). When the gravitational term is neglected the above equation became

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial z}\left(D(\theta)\frac{\partial\theta}{\partial z}\right) \tag{19}$$

The last equation has the same form as equation (12).
The diffusivity form for Darcy- Buckingham equation is

$$q=-D(\theta)\frac{\partial\theta}{\partial z}+K(\theta) \tag{20}$$

The main reason for the use of differential equation in diffusivity form is given by the reduction of the number of variable.

CONCLUSIONS

Due to peculiar condition of each experiment that we study in most of the cases the model use one of the equations mention above. There is some particular type of soil that works with other mathematical model (e. g. [1]). The soil scientists prefer the Darcy or Richard equations and mathematicians work with diffusion equation. For both sides remains the true problem: finding the solution of these equations. In order to find the solution is mandatory that equation satisfy some initial (Dirichlet problem) or boundary condition (Newman problem), or both conditions. We mention some initial and boundary condition:

- initial conditions – the values for θ and h for all z. When the initial condition demands $q=0\Rightarrow\frac{dH}{dz}=0$ along the entire column. If the soil water content, θ_i, is constant with depth, a flux corresponding to the unit gradient of H exits. If θ_i is very small the downward flux may be very small.
- boundary conditions. A boundary for the 1-dimensional problem is the topographical surface; the other boundary can be a water table if the soil column has a finite length or a defined water content or water flux. If the column is semi-infinite the other boundary is when $z\to\infty$.

Even for those cases is not easy to solve this differential equations, sometimes is possible to find a analytical solutions in the other it is possible to solve the problem numerically [2].

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